Sensitivity Analysis for Convex Multiobjective Programming in Abstract Spaces

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The main object of this paper is to prove that for a linear or convex multiobjective program, a dual program can be obtained which gives the primal sensitivity without any special hypothesis about the way of choosing the optimal solution in the efficient set.

INTRODUCTION

As is well known, many authors have studied the properties of multiobjective optimization problems including the existence and determination of the solutions and the properties of these solutions. Consequently, important methods of resolution have been found by the use of a wide class of techniques and interesting dual programs have been obtained which characterize the primal solutions. The problem of analyzing the sensitivity of the program with respect to the changes in the vector of the right side has also been studied by several authors (see, for instance, [13, 14] for the linear multiobjective programming, and [10, 11] for the nonlinear case). In this paper, we prove that for a convex multiobjective program, a dual program can be obtained which gives the primal sensitivity without any special hypothesis about the way of choosing the optimal solution in the efficient set.

The present work begins by introducing several concepts, such as the concepts of T-optimal solution and T-dual program, and then later defining a dual program for convex programming, which extends the dual programs stated in [4, 13] (for linear programming) and [17] (for convex scalar programming). After introducing the concept of associated solutions, Theorem 10 proves their existence and subsequently Theorem 12 and 15 prove the surprising fact that, in the multiobjective case, the sensitivity of the program is measured by the dual solution plus the derivative of this solution or a projection of such derivative, pointing out that this result is a generalization of the scalar case.

The comparative advantage of operators T taking values in any ordered Banach space of finite or infinite dimension is that the generalization can have, as a consequence, important properties for some special classes of programs for which Theorems 12 and 15 imply a much more restricted range where the derivative of the dual solution can be valued. As a special case of operator T, we consider the operator introduced in Theorem 5 of [5], which is a topological isomorphism and then the projection onto Ker T of the derivative of the dual solution is null. It may also be viewed as a linear and continuous mapping in the real line, with a duality theory for proper optimums useful in performing their sensitivity analysis.

T-OPTIMAL SOLUTIONS AND LAGRANGIAN T-MULTIPLIERS

Let X, Y, Z, and W be four Banach spaces and let us assume that Y, Z, and W are ordered vector spaces with positive cones Y_+, Z_+, and W_+, respectively, being the orders of Y and W antisymmetrical and the order of W verifies the infimum axiom (i.e., for each non-empty order-bounded from below subset B of W the inf B exists, that is, there exists $w_0 \in W$ such that $w_0 \leq w$ for every $w \in B$ and if there is $w_1 \in W$ verifying the $w_1 \leq w$ for every $w \in B$, then $w_1 \leq w_0$). From now on, the cones Y_+, Z_+, and W_+ will be closed and Z_+ and W_+ will have moreover non-empty interior (i.e., $Z_+^\circ \neq \varnothing$ and $W_+^\circ \neq \varnothing$).

Let $T\colon Y \to W$ be a linear and continuous surjective mapping such that $T(Y_+ - \{0\}) \subset W_+ - \{0\}$, and let us denote by D a convex subset of X and by $f\colon D \to Y$ and $g\colon D \to Z$ two convex functions.

Consider now the program

$$\left.\begin{array}{l} \operatorname{Min} f(x) \\ x \in D_{g-b} \end{array}\right\} \qquad (1_{g-b})$$

with $D_{g-b} = \{x \in D: g(x) \leq b\}$ and $b \in Z$. In the particular case of being $b = 0$ we will write (1_g) and D_g.

DEFINITION 1. A point $x_0 \in D_g$ is said to be a T-optimal solution (of the program (1_g)) if $Tf(x_0) \leq Tf(x)$ holds for every $x \in D_g$ (we will write Tf instead of $T \circ f$, TL instead of $T \circ L$, and so on). It is clear that every T-optimal solution of (1_g) is an optimal solution of that program.

We say that $y \in Y$ is a T-ideal point (of the program (1_g)) if

$$T(y) = \inf\left\{Tf(x): x \in D_g\right\}. \tag{1.1}$$

Trivially, $x_0 \in D$ is a T-optimal solution of (1_g) if and only if $f(x_0)$ is a T-ideal point of this program, and it is said that the program (1_g) is T-bounded if it admits one (or more) T-ideal point, which is equivalent to the order-boundedness from below of the subset $\{Tf(x): x \in D_g\}$ of W.

DEFINITION 2. A positive (i.e., preserving the order) linear and continuous mapping $L: Z \to W$ (we will write $L \in \mathscr{L}_+(Z, W)$) is said to be a Lagrangian T-multiplier (for the program (1_g)) if the following equality holds:

$$\inf\left\{Tf(x): x \in D_g\right\} = \inf\{Tf(x) + Lg(x): x \in D\}.^{1} \tag{2.1}$$

THEOREM 3. *If the program (1_g) has a T-optimal solution, there exists an $x_1 \in D_g$ such that $g(x_1) \in -Z_+^\circ$ and the order of Y verifies the infimum axiom, then there exists a Lagrangian T-multiplier for the program (1_g).*

Proof. It follows from Theorem 5 of [28] that there exists $(x_0, L_0) \in D_g \times \mathscr{L}_+(Z, W)$ such that the inequalities

$$Tf(x) + L_0 g(x) \geq Tf(x_0) + L_0 g(x_0) \geq Tf(x_0) + Lg(x_0)$$

hold for every $x \in D$ and every $L \in \mathscr{L}_+(Z, W)$. So taking L the null operator from Z into W we obtain that

$$\begin{aligned} \operatorname{Inf}\left\{Tf(x): x \in D_g\right\} &\geq \operatorname{Inf}\{Tf(x) + L_0 g(x): x \in D\} \\ &\geq Tf(x_0) + L_0 g(x_0) \\ &\geq Tf(x_0), \end{aligned}$$

from where it follows immediately (see footnote 1) that L_0 is a Lagrangian T-multiplier for the program (1_g).

[1] Remark that the inequality $\inf\{Tf(x) + Lg(x): x \in D\} \leq \inf\{Tf(x): x \in D_g\}$ is always satisfied for every $L \in \mathscr{L}_+(Z, W)$.

THEOREM 4. *If* $b \in Z$, $y_0, y_b \in Y$ *are two T-ideal points of the programs* (1_g) *and* (1_{g-b}), *respectively,* $L_0, L_b \in \mathscr{L}_+(Z, W)$ *are two Lagrangian T-multipliers for the programs* (1_g) *and* (1_{g-b}), *respectively, then*

$$-L_b(b) \geq T(y_b) - T(y_0) \geq -L_0(b). \tag{4.1}$$

In the particular case of being $x_0 \in D_g$ *and* $x_b \in D_{g-b}$ *two T-optimal solutions for the programs* (1_g) *and* (1_{g-b}), *respectively, then we have that*

$$-L_b(b) \geq Tf(x_b) - Tf(x_0) \geq -L_0(b). \tag{4.2}$$

Proof. This is an immediate consequence of Definitions 1 and 2 since for every $x \in D_g$ and every $x' \in D_{g-b}$ we have that

$$T(y_b) \leq Tf(x) + L_b(g(x) - b) \leq Tf(x) - L_b(b)$$

and

$$T(y_0) \leq Tf(x) + L_0 g(x') \leq Tf(x') + L_0(b),$$

from where the result follows trivially.

THEOREM 5. *If W is a Banach lattice and there exists a neighborhood V of the zero vector of Z such that for every* $b \in V$ *there exists a T-optimal solution* $x_b \in D_{g-b}$ *and a Lagrangian T-multiplier* $L_b \in \mathscr{L}_+(Z, W)$, *both for the program* (1_{g-b}), *such that* $\lim_{b \to 0} L_b = L_0$ *in the space* $\mathscr{L}(Z, W)$ *of the linear and continuous mappings from Z into W, endowed with the topology of the uniform convergence on the bounded subsets of Z, then the function* $F\colon V \to W$, *such that* $F(b) = Tf(x_b)$ *for every* $b \in V$, *is Fréchet differentiable at* $0 \in V$ *and its Fréchet differential at zero coincides with* $-L_0$.

Proof. For every $b \in V - \{0\}$, it follows from (4.2) that

$$0 \leq \frac{Tf(x_b) - Tf(x_0) + L_0(b)}{\|b\|} \leq \frac{L_0(b) - L_b(b)}{\|b\|}$$

from where the result is immediately deduced since

$$\frac{\|L_0(b) - L_b(b)\|}{\|b\|} \leq \|L_0 - L_b\|.$$

THE DUAL PROGRAM AND ASSOCIATED SOLUTIONS

From now on, let us denote by $\mathscr{T}$ the set of the surjective linear and continuous mappings $T\colon Y \to W$ such that $T(Y^+ - \{0\}) \subset W^+ - \{0\}$ and $\operatorname{Ker} T$ has a topological supplement in Y, and for every $T \in \mathscr{T}$ let Γ_T be the set of all the operators $L \in \mathscr{L}_+(Z, W)$ such that $\{Tf(x) + Lg(x): x \in D\}$ is an order-bounded from below subset of W. For every $T \in \mathscr{T}$ and $L \in \Gamma_T$ let us consider

$$\varphi(T, L) = \inf\{Tf(x) + Lg(x): x \in D\} \in W.$$

If $T \in \mathscr{T}$ and Y_T is a topological supplement in Y of Ker T then, since Y_T is closed, it follows from the open-mapping theorem that the restriction $\hat{T}$ of T to Y_T is an isomorphism from Y_T onto W. For simplifying the notation we will suppose fixed Y_T for every $T \in \mathscr{T}$ (in the case of Y being a Hilbert space, Y_T could be for instance (the orthogonal of Ker T) $(\text{Ker } T)^\perp$).

Finally, for every $T \in \mathscr{T}$ and $L \in \Gamma_T$ let

$$\psi(T, L) = \hat{T}^{-1}\varphi(T, L) \in Y.$$

DEFINITION 6. Let $b \in Z$ and the program

$$\left.\begin{array}{r} \text{Min } f(x) \\ x \in D \\ g(x) \leq b \end{array}\right\}. \tag{1_{g-b}}$$

For every $T \in \mathscr{T}$, the T-dual program of the program (1_{g-B}) will be

$$\left.\begin{array}{l} \text{Max } \varphi(T, L) - L(b) \\ L \in \Gamma_T \end{array}\right\}. \tag{2_{g-b}}$$

Moreover, the dual program of the program (1_{g-b}) will be

$$\left.\begin{array}{c} \text{Max } \psi(T, TG) - G(b) \\ T \in \mathscr{T} \\ G \in \mathscr{L}(Z, Y) \\ TG \in \Gamma_T \end{array}\right\}. \tag{3_{g-b}}$$

PROPOSITION 7. *If $x \in D_{g-b}$, $G \in \mathscr{L}(Z, Y)$, $T \in \mathscr{T}$, and $TG \in \Gamma_T$, then*

$$\psi(T, TG) - G(b) - f(x) \notin Y_+ - \{0\}, \tag{7.1}$$

which means that the dual objective is never greater than the primal one.

Proof. If (7.1) does not hold then we would have that

$$\varphi(T, TG) - TG(b) - Tf(x) \in W^+ - \{0\}$$

and, therefore,

$$Tf(u) + TG[g(u) - b] - Tf(x) \in W^{+} - \{0\}$$

for every $u \in D$, and in the particular case of being $u = x$ we have that

$$TG[g(x) - b] \in W^{+} - \{0\}. \tag{7.2}$$

Since $x \in D_{g-b}$ and $TG \in \Gamma_T \subset \mathscr{L}_{+}(Z, W)$, it results that

$$TG[g(x) - b] \in -W^{+}, \tag{7.3}$$

and now it follows from (7.2) and (7.3) a contradiction.

PROPOSITION 8. *If $x \in D_{g-b}$, $G \in \mathscr{L}(Z, Y)$, $T \in \mathscr{T}$, $TG \in \Gamma_T$, and $\psi(T, TG) - G(b) = f(x)$, then x is an optimal solution of (1_{g-b}) and (T, G) is an optimal solution of (3_{g-b}).*

Proof. This is an immediate consequence of Proposition 7.

DEFINITION 9. Under the notations of Proposition 8, x and (T, G) will be called associated solutions.

THEOREM 10. *Let $T \in \mathscr{T}$ and $x_0 \in D_{g-b}$ be a T-optimal solution of the program (1_{g-b}). If $b \neq 0$ then the following assertions are equivalent*:

10.1 *There exists $L \in \mathscr{L}_{+}(Z, W)$ which is a Lagrangian T-multiplier for the program (1_{g-b}).*

10.2 *There exists $G \in \mathscr{L}(Z, Y)$ such that (T, G) is an optimal solution of the program (3_{g-b}) and x_0 and (T, G) are associated solutions.*[2]

Proof. First let us suppose that 10.1 holds, then

$$Tf(x_0) = \inf\{Tf(x) + L[g(x) - b] : x \in D\} \tag{10.1}$$

and since Y_T is a topological supplement in Y of Ker T, then there exists $y_0 \in \operatorname{Ker} T$ verifying that

$$f(x_0) = y_0 + \hat{T}^{-1}[Tf(x_0)]. \tag{10.2}$$

Moreover, since $b \neq 0$ we can find $z' \in Z'$ (Z' being as usual the dual space of Z) such that

$$z'(b) = 1. \tag{10.3}$$

[2] Let us remark that (10.2) implies (10.1) also being $b = 0$, and moreover that $L = TG$. Also it must be pointed out that the proof of the implication (10.1) $\Rightarrow$ (10.2) is a constructive one.

Let us define

$$G(z) = \hat{T}^{-1}[L(z)] - z'(z)y_0$$

for every $z \in Z$. Clearly, $G \in \mathscr{L}(Z, y)$ and $TG = L \in \mathscr{L}_+(Z, W)$, since $y_0 \in \operatorname{Ker} T$. Moreover we have that

$$\begin{aligned}\varphi(T, TG) &= \varphi(T, L) \\ &= \inf\{Tf(x) + Lg(x) : x \in D\} \\ &= \inf\{Tf(x) + L[g(x) - b] : x \in D\} + L(b),\end{aligned}$$

and then it follows from (10.1) that $\varphi(T, TG) = Tf(x_0) + L(b)$. Therefore, $TG \in \Gamma_T$ and

$$\begin{aligned}\psi(T, TG) &= \hat{T}^{-1}[\varphi(T, TG)] \\ &= \hat{T}^{-1}[Tf(x_0)] + \hat{T}^{-1}L(b),\end{aligned}$$

and it follows from (10.3) that $G(b) = \hat{T}^{-1}L(b) - y_0$ and

$$\psi(T, TG) = \hat{T}^{-1}Tf(x_0) + G(b) + y_0,$$

so it results from (10.2) that

$$\begin{aligned}\psi(T, TG) - G(b) &= \hat{T}^{-1}Tf(x_0) + y_0 \\ &= f(x_0).\end{aligned}$$

Let us assume now that 10.2 holds and let be $L = TG \in \mathscr{L}_+(Z, W)$. Then since $f(x_0) = \psi(T, TG) - G(b)$, we have that

$$\begin{aligned}Tf(x_0) &= \varphi(T, TG) - L(b) \\ &= \inf\{Tf(x) - Lg(x) : x \in D\} - L(b) \\ &= \inf\{Tf(x) - L[g(x) - b] : x \in D\}\end{aligned}$$

and L is a Lagrangian T-multiplier for the program (1_{g-b}).

SENSITIVITY ANALYSIS

LEMMA 11. *Let V be an open subset of Z, M a Banach space, and $\mathscr{H}: V \to \mathscr{L}(Z, M)$ a Fréchet differentiable function ($\mathscr{L}(Z, M)$ denotes as usual the space of the linear and continuous mappings from Z into M, endowed with the topology of the uniform convergence on the bounded subsets* *of Z). If $F\colon V \to M$ is such that $F(b) = H_b(b)$ with $H_b = \mathscr{H}(b)$, for every $b \in V$, then F is Fréchet differentiable at V and*

$$F'(b, z) = H_b(z) + \mathscr{H}'(b, z)(b)$$

for every $z \in Z$ and every $b \in V$.[3]

Proof. In fact, if $b \in V$ and $z \in Z$ then we have that

$$\frac{1}{\|z\|}\|F(b+z) - F(b) - H_b(z) - \mathscr{H}(b,z)(b)\|$$

$$= \frac{1}{\|z\|}\|H_{b+z}(b+z) - H_b(b) - H_b(z) - \mathscr{H}(b,z)(b)\|$$

$$\leq \frac{1}{\|z\|}\left[\|H_{b+z}(b) - H_b(b) - \mathscr{H}'(b,z)(b)\| + \|H_{b+z}(z) - H_b(z)\|\right],$$

from where the result follows immediately since the function $\mathscr{H}$ is Fréchet differentiable at $b \in V$ and, therefore, it is also continuous at $b \in V$.

We are now able to prove a rather surprising result. In general, in a multiobjective convex program, the sensitivity of the optimum depends not only on the value of the dual solution but also on the differential of this dual solution and more concretely, on the projection of such a derivative onto Ker T. In the scalar programming cases (i.e., $Y = \mathbb{R}$), such a projection is clearly null and therefore, the next Theorem 12 is a generalization of the corresponding happening in the scalar programming.

As we will see later on, in the particular case of the linear programming the differential of the dual solution is in Ker T (and therefore, it coincides with its projection onto Ker T).

THEOREM 12. *Let W be a Banach lattice, V an open subset of Z, $T \in \mathscr{T}$, $x_b \in D_{g-b}$ a T-optimal solution of the program (1_{g-b}), and $G_b \in \mathscr{L}(Z,Y)$ an operator such that (T, G_b) is an optimal solution of the program (3_{g-b}) associated with x_b, for every $b \in V$. If the function $\mathscr{G}\colon V \to \mathscr{L}(Z,Y)$ defined by $\mathscr{G}(b) = G_b$ for every $b \in V$, is Fréchet differentiable (at V) then the function $F\colon V \to Y$ such that $F(b) = f(x_b)$ for every $b \in V$, is also Fréchet differentiable (at V) and moreover, the equality*

$$F'(b,z) = -G_b(z) - K(b,z)$$

holds for every $z \in Z$ and $b \in V$, where $K(b,z)$ denotes the projection of $\mathscr{G}'(b,z)(b)$ onto Ker T, *for every $z \in Z$ and $b \in V$.*

[3] $F'(b,z)$ denotes the image of $z \in Z$ by the Fréchet differential of the function F at $b \in V$. A similar notation is followed in the other cases.

Proof. For every $b \in V$ let $L_b = TG_b \in \Gamma_T$. It follows from the proof of Theorem 10 that L_b is a Lagrangian T-multiplier for the program (1_{g-b}). Consider now the functions $\varphi^*: V \to W$ and $\psi^*: V \to Y$ such that $\varphi^*(b) = \varphi(T, L_b)$ and $\psi^*(b) = \psi(T, TG_b)$ for every $b \in V$. Since (T, G_b) and x_0 are associated solutions we have that

$$F(b) = f(x_b) = \psi(T, TG_b) - G_b(b) \tag{12.1}$$

and

$$Tf(x_b) = \varphi(T, L_b) - L_b(b)$$

for every $b \in V$, and it results from Theorem 5 and Lemma 11 that

$$-L_b(z) = (\varphi^*)'(b,z) - L_b(z) - \mathscr{L}'(b,z)(b) \tag{12.2}$$

for every $z \in Z$ and $b \in V$, where $\mathscr{L}: V \to \mathscr{L}(Z,V)$ is the function defined by $\mathscr{L}(b) = L_b$ for every $b \in V$. Clearly, the equality (12.2) is equivalent to

$$(\varphi^*)'(b,z) = \mathscr{L}'(b,z)(b) \tag{12.3}$$

for every $z \in Z$ and $b \in V$, and since $\varphi^* = T\psi^*$, it follows from (12.3) that

$$T(\psi^*)'(b,z) = \mathscr{L}'(b,z)(b)$$

for every $z \in Z$ and $b \in V$. Moreover, since $\mathscr{L}(b) = T\mathscr{G}(b)$ for every $b \in V$, it is immediately deduced that

$$\mathscr{L}'(b,z)(b) = T\mathscr{G}'(b,z)(b),$$
$$T(\psi^*)'(b,z) = T\mathscr{G}'(b,z)(b)$$

and

$$(\psi^*)'(b,z) = \hat{T}^{-1}T\mathscr{G}'(b,z)(b)$$

for every $z \in Z$ and $b \in V$.

Finally, from (12.1) and Lemma 11 it results that

$$\begin{aligned} F'(b,z) &= -G_b(z) - \mathscr{G}'(b,z)(b) + \hat{T}^{-1}T\mathscr{G}'(b,z)(b) \\ &= -G_b(z) - K(b,z) \end{aligned}$$

for every $z \in Z$ and $b \in V$.

EXAMPLE. To illustrate easily the result stated in Theorem 12 let us consider the program (1_{g-b}) with $b \in (-\infty, 0)$, $X = Y = \mathbb{R}^2$, $Z = W = \mathbb{R}$, $f(x_1, x_2) = (x_1^2, 2x_2^2)$, and $g(x_1, x_2) = T(x_1, x_2) = x_1 + x_2$. Solving in the usual way the program

$$\left.\begin{aligned} \text{Min } x_1^2 + 2x_2^2 \\ x_1 + x_2 \leq b \end{aligned}\right\}$$

the solution $x_b = (\frac{2}{3}b, \frac{1}{3}b)$ and the Lagrangian T-multiplier $L_b = -\frac{4}{3}b$ are obtained. Therefore, the function F: $(-\infty, 0) \to \mathbb{R}^2$ (of Theorem 12) is defined by

$$F(b) = f\left(\tfrac{2}{3}b, \tfrac{1}{3}b\right) = \left(\tfrac{4}{9}b^2, \tfrac{2}{9}b^2\right)$$

and then

$$F'(b) = \left(\tfrac{8}{9}b, \tfrac{4}{9}b\right).$$

On the other hand, Ker T is the linear space generated by the vector $(1, -1)$ and so (Ker T) will be the linear subspace generated by $(1, 1)$.

Following the construction made in the proof of Theorem 10, we obtain

$$\hat{T}^{-1}[L_b] = \left(-\tfrac{2}{3}b, -\tfrac{2}{3}b\right)$$

and

$$\begin{aligned} y_b &= f(x_b) - \hat{T}^{-1}[Tf(x_b)] \\ &= \left(\tfrac{4}{9}b^2, \tfrac{2}{9}b^2\right) - \hat{T}^{-1}T\left(\tfrac{4}{9}b^2, \tfrac{2}{9}b^2\right) \\ &= \left(\tfrac{4}{9}b^2, \tfrac{2}{9}b^2\right) - \hat{T}^{-1}\left(\tfrac{6}{9}b^2\right) \\ &= \left(\tfrac{4}{9}b^2, \tfrac{2}{9}b^2\right) - \left(\tfrac{3}{9}b^2, \tfrac{3}{9}b^2\right) \\ &= \left(\tfrac{1}{9}b^2, -\tfrac{1}{9}b^2\right). \end{aligned}$$

Taking now $z_b'(z) = (1/b)z$ for every $z \in \mathbb{R}$, we have that

$$G_b = \hat{T}^{-1}[L_b] - \frac{1}{b}y_b = \left(-\tfrac{7}{9}b, -\tfrac{5}{9}b\right).$$

Therefore, since the projection onto Ker T is the function π: $\mathbb{R}^2 \to$ Ker T such that

$$\pi(x_1, x_2) = \left(\frac{x_1 - x_2}{2}, -\frac{x_1 - x_2}{2}\right)$$

for every $(x_1, x_2) \in \mathbb{R}^2$, with the notations of Theorem 12, we have that

$$K(b) = \pi\mathscr{G}'(b)(b) = \left(-\frac{b}{9}, \frac{b}{9}\right)$$

and therefore,

$$-G_b - K(b) = \left(\tfrac{8}{9}b, \tfrac{4}{9}b\right)$$

and $-G_b - K(b) = F'(b)$, as Theorem 12 states.

DUALITY AND SENSITIVITY FOR LINEAR PROGRAMS

Let us now treat the important particular case of linear programs. For them, the dual program has a much simpler formulation than that given by Definition 6 (as we will see in Theorem 14). Of special interest is Theorem 15 which is analogous to Theorem 12, but which states that the term $\mathscr{G}'(b, z)(b)$ belongs now to $\operatorname{Ker} T$ and, therefore, it coincides with its projection.

Suppose henceforth that D is a (non-necessarily pointed) convex cone of X and that $f \in \mathscr{L}(X, Y)$ and $g \in \mathscr{L}(X, Z)$. Then the spaces $\mathscr{L}(X, Y)$ and $\mathscr{L}(X, Z)$ will be assumed to be ordered by its natural cones

$$\mathscr{L}_+(X, Y) = \{\xi \in \mathscr{L}(X, Y) \colon \xi(x) \in Y_+ \text{ for every } x \in D\},$$
$$\mathscr{L}_+(X, Z) = \{\xi \in \mathscr{L}(X, Z) \colon \xi(x) \in Z_+ \text{ for every } x \in D\}$$

and the same will be for $\mathscr{L}(X, W)$.

LEMMA 13. *Under the already established notations, if $T \in \mathscr{T}$ then*

$$\Gamma_T = \{L \in \mathscr{L}_+(Z, W) \colon Lg \geq -Tf\}^4$$

and $\varphi(T, L) = 0$ for every $L \in \Gamma_T$.

Proof. Let $L \in \mathscr{L}_+(Z, W)$. If $Lg \geq -Tf$, then $Lg(x) + Tf(x) \in W_+$ for every $x \in D$ and, therefore, $\{Tf(x) + Lg(x) \colon x \in D\} \subset W_+$ is order-bounded from below. Moreover, since $0 \in D$ we have that

$$\varphi(T, L) = \inf\{Tf(x) + Lg(x) \colon x \in D\} = 0.$$

If $L \in \Gamma_T$ then $\{Tf(x) + Lg(x) \colon x \in D\}$ is an order-bounded from below subset of W and, therefore, $\{w'[TF(x) + Lg(x)] \colon x \in D\}$ is a bounded

[4] $Lg \geq -Tf$ means as usual that $Lg(x) + Tf(x) \in W_+$ for every $x \in D$.

from below subset of $\mathbb{R}$ for every $w' \in W'_+$, where W'_+ denotes as usual the dual cone of W_+ in the dual space W' of W, and therefore, we have that

$$w'[TF(x) + Lg(x)] \geq 0 \tag{13.1}$$

for every $w' \in W'_+$ and $x \in D$ (since if there exists $w'_0 \in W'_+$ and $x_0 \in D$ such that $w'_0[TF(x_0) + Lg(x_0)] < 0$ then

$$\lim_{\alpha \to +\infty} w'_0[Tf(\alpha x_0) + Lg(\alpha x_0)] = -\infty,$$

which is a contradiction).

Now since $W_+^\circ \neq \emptyset$, it follows from (13.1) that $Tf(x) + Lg(x) \in W_+$ for every $x \in D$ (see [12]).

THEOREM 14. *The dual program of* (1_{g-b}) (*under the present assumption of* (1_{g-b}) *being a linear program*) *is*

$$\left.\begin{array}{r} \operatorname{Max} - G(b) \\ T \in \mathscr{T} \\ G \in \mathscr{L}(Z,Y) \\ TGg \geq -Tf \\ TG \in \mathscr{L}_+(Z,W) \end{array}\right\}. \tag{4_{g-b}}$$

Proof. This follows immediately from Definition 6 and Lemma 13.

THEOREM 15. *Let W be a Banach lattice, $T \in \mathscr{T}$, and an open subset V of Z such that for every $b \in V$ there exists $x_b \in V$ and $G_b \in \mathscr{L}(Z,Y)$ such that x_b is a T-optimal solution of (1_{g-b}), (T, G_b) is an optimal solution of (4_{g-b}), and x_b and (T, G_b) are associated solutions. Suppose that the mappings $\mathscr{G}\colon V \to \mathscr{L}(X,Y)$ such that $\mathscr{G}(b) = G_b$ for every $b \in V$, is Fréchet differentiable (at V) and let $F\colon V \to Y$ be the function defined by $F(b) = f(x_b)$ for every $b \in V$. Then F is Fréchet differentiable (at V) and the equalities*

$$F'(b,z) = -G_b(z) - \mathscr{G}'(b,z)(b) \tag{15.1}$$

and

$$T[\mathscr{G}'(b,z)(b)] = 0 \tag{15.2}$$

hold for every $z \in Z$ and $b \in V$.

Proof. It follows from Theorem 12 that the function F is Fréchet differentiable (at V) and that

$$F'(b,z) = -G_b(z) - K(b,z) \tag{15.3}$$

holds for every $z \in Z$ and $b \in V$, where $K(b, z)$ is the projection of $\mathscr{G}'(b, z)(b)$ onto Ker T. Moreover, since x_b and (T, G_b) are associated solutions for every $b \in V$, it results from Theorem 14 that

$$F(b) = f(x_b) = -G_b(b)$$

for every $b \in V$, and, therefore, (15.1) follows from Lemma 11.

Moreover, it is immediately deduced from (15.1) and (15.3) that $\mathscr{G}'(b, z)(b) = K(b, z)$ for every $z \in Z$ and $b \in V$ and, therefore, (15.2) is verified.

CONCLUSIONS

The dual program stated here for convex multiobjective programming extends the dual programs given in [4, 13] (for linear multiobjective programming) and [17] (for convex scalar programming). Theorem 10 proves the existence of associated solutions which are introduced in Definition 9 and Theorems 12 establishes the surprising fact that, in the multiobjective case, the sensitivity of the program is measured by the dual solution plus the derivative of this solution or a projection of such a derivative, pointing out that this result is a generalization of the scalar case. As an important particular case linear programs are also studied since for them the results have a simpler formulation. The theory developed here in the context of Banach spaces is quite general and it may be applied in many particular situations like static, dynamic, or semi-infinite programs.

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